Differential Equations, Infinite Limits and Real Recursive Functions

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Abstract: In this article we present a strong support to real recursive function theory as a branch of computability theory rooted in mathematical analysis. This new paradigm connects computation on reals with differential equations and infinite limits in a robust and smooth way. The results presented here are taken mainly from the article (4) of the same authors.

Key–Words: Computation on Reals, Differential Equations, Infinite Limits, Analytical Hierarchy, Decidability

1 Introduction

The notion of algorithm was clarified by Alan Turing, who gave it a precise meaning as well as introduced a coherent framework for discrete computation. Many results showing the relations of his model with other approaches, such as recursive functions or Church's λ-calculus (compare (7)), led to a consistent theoretical basis for standard computation theory. However, all these models use enumerable domains and treat time of computations as discrete.

Nevertheless, computers can be analog devices. Analog computers with the continuous internal states rather than discrete, as in digital computation, were invented and discussed quite thoroughly (9; 8). Unfortunately, because of the problem of a good theoretical basis for analog computation and lack of improvements in analog computers technology (compared with its digital counterpart), analog computation was about to be forgotten. For many reasons (new paradigms of computation, search for good tools for numerical analysis, new technologies) this situation seems to change nowadays (3; 1).

In 1996 Cris Moore published a seminal paper (5), where he defined an inductive class of vector valued functions over $\mathbb{R}$. This class was defined as the closure of some basic functions for the operators of composition, solving of first-order differential equations and a kind of minimalization. Unfortunately, some of Moore's assumptions were not accepted for their lack of mathematical precision. Most of these controversial assumptions were consequences of Moore's attempt to bring the minimalization operator – used in the classical recursive functions – into a continuous context. So in (6) the modified definition was given replacing minimalization with the taking of infinite limits. Here we use this definition in the more sophisticated form, which was formulated in (4) and we present some discussion of main results forming the basis for the robust theory of real recursive functions.

2 Definitions and properties

Let us introduce a few operators useful in this paper. We should start with the operator of differential recursion. For this purpose let us recall some standard notions and properties from mathematical analysis.

Definition 1 *A total function $f : \mathbb{R}^m \to \mathbb{R}^n$ is called locally Lipschitz if for every compact set $C \subset \mathbb{R}^m$ there is a constant K such that all $\vec{x}, \vec{y} \in C$ verify the Lipschitz condition*

$$\|f(\vec{x}) - f(\vec{y})\| \leq K\|\vec{x} - \vec{y}\|. \quad (1)$$

The smallest such K is called the Lipschitz constant of f for C.

Notice that the concept of Lipschitz constant is well-defined, by the compactness of C and the continuity of f and of the Euclidean norm. The local

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Lipschitz property implies continuity. The name *locally Lipschitz* is motivated because a total function $f : \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz if and only if around every point $\vec{z} \in \mathbb{R}^m$ there is a neighborhood V of $\vec{z}$ and a constant K such that all $\vec{x}, \vec{y} \in V$ satisfy (1).

Now we can start building our class of real recursive functions. We take $\mathscr{F}$ to be the class of partial, vector-valued, multiple argument functions over $\mathbb{R}$, i.e., the class of partial functions $f : \mathbb{R}^m \to \mathbb{R}^n$ for some $m, n \in \mathbb{N}$. We accept functions of arity 0 and call them constants.

There will be two kinds of basic functions: the constant functions, and the projections. The constant functions are denoted -1^n, 0^n, and 1^n, for every $n = 0, 1, 2, \ldots$, and are given by $-1^n(x_1, \ldots, x_n) = -1$, $0^n(x_1, \ldots, x_n) = 0$ and $1^n(x_1, \ldots, x_n) = 1$. The projections are denoted with U_i^n, for each $n = 1, 2, \ldots$ and $1 \leq i \leq n$; they are given by $\mathrm{U}_i^n(x_1, \ldots, x_n) = x_i$.

The class will be closed under some number of partial operators over $\mathscr{F}$.

The first operator is the composition operator denoted by $\mathbf{C}$. Given two functions $f : \mathbb{R}^k \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^k$, the function $\mathbf{C}(f, g)$ goes from $\mathbb{R}^m$ to $\mathbb{R}^n$, and is given by

$$\mathbf{C}(f, g)(\vec{x}) = f(g(\vec{x})), \text{ for every } \vec{x} \in \mathbb{R}^m.$$

The domain of $\mathbf{C}(f, g)$ is $Dom(\mathbf{C}(f, g)) = \{\vec{x} \in Dom(g) : g(\vec{x}) \in Dom(f)\}$

Our second operator is the differential recursion operator, denoted by $\mathbf{R}$. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a total locally Lipschitz function. Consider, for a fixed $\vec{x} \in \mathbb{R}^m$, the Cauchy problem

$$g(\vec{x}, 0) = \vec{x} \quad \partial_t g(\vec{x}, t) = f(t, g(\vec{x}, t)). \tag{2}$$

Then $\mathbf{R}(f)$ is a function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. For every fixed $\vec{x} \in \mathbb{R}^n$, $\mathbf{R}(f)(\vec{x}, t) = g(\vec{x}, t)$, where $g(\vec{x}, \cdot)$ is the maximal solution of (2) — i.e. the function g defined in (A, B) such that if $\tilde{g}$ is a solution of (2) over some interval (a, b), then $g(t) = \tilde{g}(t)$ for all $t \in (a, b)$. If the interval (A, B) is called the maximal interval of (2) then the domain of $\mathbf{R}(f)$ is $Dom(\mathbf{R}(f)) = \{(\vec{x}, t) : \vec{x} \in \mathbb{R}^m, A(\vec{x}) < t < B(\vec{x})\}$, where A, B give the extrema of the maximal interval.

The next operator is the infinite supremum limit operator, denoted by $\mathbf{Ls}$. This operator takes any function $f : \mathbb{R}^{m+1} \to \mathbb{R}^n$, and maps it into the component-wise infinite supremum limit, i.e., for every $i = 1, \ldots, n$, $(\mathbf{Ls}(f)(\vec{x}))_i = \limsup_{y \to \infty} (f(\vec{x}, y))_i$. For the sake of abbreviation, we write simply

$$\mathbf{Ls}(f)(\vec{x}) = \limsup_{y \to \infty} f(\vec{x}, y).$$

Then $Dom(\mathbf{Ls}(f)) = \{\vec{x} \in \mathbb{R}^m : \limsup_{y \to \infty} f(\vec{x}, y) \text{ exists}\}$.

The last operator is called the aggregation operator, denoted by the symbol $\mathbf{V}$. The aggregation operator takes two functions $f : \mathbb{R}^m \to \mathbb{R}^k$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ and joins them into a single vector function $\mathbf{V}(f, g) : \mathbb{R}^m \to \mathbb{R}^{k+n}$. As expected, this is given by

$$\mathbf{V}(f, g)(\vec{x}) = (f(\vec{x}), g(\vec{x})),$$

and $Dom(\mathbf{V}(f, g)) = Dom(f) \cap Dom(g)$.

Using the notation of function algebras we can finally give the main definition.

Definition 2 *The class of real recursive functions, denoted by* $\mathrm{REC}(\mathbb{R})$*, is given by the function algebra*

$$\mathrm{REC}(\mathbb{R}) = [-1^n, 0^n, 1^n, \mathrm{U}_i^n; \mathbf{C}, \mathbf{R}, \mathbf{Ls}, \mathbf{V}].$$

Let us give a short description of properties of the above introduced operators. We start pointing out good behavior of differential recursion.

Theorem 3 *For any* $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *in the domain of* $\mathbf{R}$, $g = \mathbf{R}(f)$ *is locally Lipschitz.*

It is worth mentioning that the Cauchy problem of more general form, such as

$$g(\vec{x}, t_0) = g_0(\vec{x}) \quad \partial_t g(\vec{x}, t) = f(t, g(\vec{x}, t), \vec{x}) \tag{3}$$

can be reduced to the form (2) used in Definition 2.

Let us add a simple example of this operator.

Example 4 *Consider the differential recursion schema*

$$g(0) = (0, 1) \qquad \partial_t g(t) = (g_2(t), -g_1(t)).$$

With the notation of (3), we have $g_0 = (0, 1)$ *and* $f(t, \vec{z}) = ((\vec{z})_2, -(\vec{z})_1)$*. The solution can be recognized as* $g = (\sin, \cos)$. □

The class $\mathrm{REC}(\mathbb{R})$ is composed of partial, multiple-argument vector functions over $\mathbb{R}$. We should give a precise characterization how the concept of infinite supremum limit can be applied to such functions. Let us observe that any partial function $f : \mathbb{R}^m \to \mathbb{R}^n$ can be uniquely identified with its graph. The graph of f, G_f is a predicate over $\mathbb{R}^{m+n}$ given by:

$$G_f(\vec{x}, \vec{z}) \iff \vec{x} \in Dom(f) \text{ and } \vec{z} = f(\vec{x}).$$

Now, let us analyze the case when $h = \mathbf{Ls}(f)$ (i.e., $h(\vec{x}) = \limsup_{y \to \infty} f(\vec{x}, y)$). We can use graphs of functions, which give the following equivalence:

$$G_h(\vec{x}, \vec{z}) \iff (\forall \varepsilon > 0)(\exists \tilde{w} > 0)(\forall w > \tilde{w}) \; \underline{\exists \vec{v}\, G_g(\vec{x}, w, \vec{v})} \wedge \|\vec{v} - \vec{z}\| < \varepsilon.$$

As we can check, the underlined sub-predicate will not be valid unless $f(\vec{x}, y)$ is total for all $y > w$. Since w is universally quantified, we obtain the following conclusion: if $f(\vec{x}, y)$ is undefined for arbitrarily large y, then $\mathbf{Ls}(f)(\vec{x})$ will be undefined, i.e., $\vec{x} \notin Dom(\mathbf{Ls}(f))$. Furthermore, in order for the supremum limit to be correctly defined we should be sure that every one of its components is defined.

Now we can give some basic theory of real recursive functions based on theorems from (6; 4).

Proposition 5 *The binary addition, subtraction and multiplication are real recursive. The restrictions to the domain $(0, +\infty)$ of the inverse, division and square root functions are real recursive. The exponential, logarithm, power, sine, cosine and arc-tangent functions are real recursive. The real numbers π and e are real recursive constants.*

Proof: Let us present only for illustration the way in which restricted division and logarithm functions are obtained through the differential recursion scheme:

$$\begin{cases} \frac{1}{1} = 1, \\ \log(1) = 1, \end{cases}$$

and

$$\begin{cases} \partial_x \frac{1}{x} = -1 \times \left(\frac{1}{x}\right) \times \left(\frac{1}{x}\right) = -\frac{1}{x^2}, \\ \partial_x \log(x) = \frac{1}{x}. \end{cases}$$

□

Some functions need to be defined with limits.

Proposition 6 *Kronecker's δ and Heaviside's Θ, given by*

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Theta(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{otherwise;} \end{cases}$$

are real recursive.

Proof: Set $\delta(x) = \limsup_{y \to \infty} \left(\frac{1}{x^2 + 1} \right)^y$ and $\Theta(x) = \left(\limsup_{y \to \infty} \frac{1}{1 + 2^{-xy}} \right) + \frac{1}{2}\delta(x)$. □

We also have the sawtooth wave function denoted by r, and the square wave function denoted by s in the class of real recursive functions. The square function is given by $s(x) = \Theta(\sin(\pi x))$. We can build the sawtooth using the recursion scheme $\tilde{r}(0) = 0$ and $\partial_x \tilde{r}(x) = 2\sin(\pi x)^2 s(x) - \frac{1}{2}$, then we put $r(x) = s(x)\tilde{r}(x+1) + (1 - s(x))\tilde{r}(x)$.

Let us add one more important result concerning iteration.

Definition 7 *The restricted iteration operator $\bar{\mathbf{I}}$, transforms an n-ary, total, locally Lipschitz function $g \in \mathscr{F}$ with n components, into a total $(n+1)$-ary function $\bar{\mathbf{I}}(g)$ with n components, given by*

$$\bar{\mathbf{I}}(g)(\vec{x}, t) = g^{\lfloor |t| \rfloor}(\vec{x}) = \underbrace{g \circ g \circ \ldots \circ g}_{\lfloor |t| \rfloor \text{ times}}(\vec{x}).$$

The following theorem proves that the iteration operator does not lead outside the class of real recursive functions.

Theorem 8 $\mathrm{REC}(\mathbb{R})$ *is closed under the iteration operator* $\bar{\mathbf{I}}$.

3 Computability on reals

In the rest of the paper we use w, x, y, z to denote variables ranging over $\mathbb{R}$, and a, b, c, i, j to denote variables ranging over $\mathbb{N}$. The corresponding vector forms $\vec{w}, \vec{x}, \ldots$ and $\vec{a}, \vec{b}, \ldots$ will denote vector-valued variables over tuples of $\mathbb{R}$ and $\mathbb{N}$.

Let us stress that many ideas of this section are rooted in (2; 4).

We start with a few simple functionals as examples: the zero functionals, where each $\mathcal{Z}^k$ is such that $\mathcal{Z}^k(x_1, \ldots, x_k; a) = 0$; the successor functionals, where each $\mathcal{S}^k$, given by $\mathcal{S}^k(x_1, \ldots, x_k; a) = a + 1$; the projection functionals, where each $\mathcal{U}_j^{k,m}$ obeys $\mathcal{U}_j^{k,m}(x_1, \ldots, x_k; a_1, \ldots, a_m) = a_j$; the oracle functionals, $\mathcal{O}_i^k$, such that $\mathcal{O}_i^k(x_1, \ldots, x_k; b) = x_i(b)$ (here $x_i(b)$ means the b-th digit of the binary expansion of x_i).

We use $\mathcal{C}$, $\mathcal{R}$ and μ to stand for the composition, primitive recursion and minimalization operators for functionals.

Given $F : \mathbb{R}^k \times \mathbb{N}^{m'} \to \mathbb{N}^n$, $G : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^{m'}$, the functional $\mathcal{C}(F, G) : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^n$ is given by

$$\mathcal{C}(F, G)(\vec{x}; \vec{a}) = F(\vec{x}; G(\vec{x}; \vec{a})).$$

For two functionals $F : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}$ and $G : \mathbb{R}^k \times \mathbb{N}^{m+2} \to \mathbb{N}$, $\mathcal{R}(F, G) : \mathbb{R}^k \times \mathbb{N}^{m+1} \to \mathbb{N}$ is given by

$$\mathcal{R}(F, G)(\vec{x}; \vec{a}, 0) = F(\vec{x}; \vec{a}),$$

$$\mathcal{R}(F, G)(\vec{x}; \vec{a}, b+1) = G(\vec{x}; b, \mathcal{R}(F, G)(\vec{x}; \vec{a}, b), \vec{a}).$$

The minimalization operator μ takes a functional $F : \mathbb{R}^k \times \mathbb{N}^{m+1} \to \mathbb{N}$ and gives $\mu(F) : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}$ such that

$$\mu(F)(\vec{x}; \vec{a}) = \min\{b \in \mathbb{N} : F(\vec{x}; \vec{a}, b) = 0\}.$$

Finally, $\mathcal{V}$ is the aggregation operator: if $F : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^n$, $F : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^k$ are two functionals, then $\mathcal{V}(F,G) : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^{n+k}$ comes from

$$\mathcal{V}(F,G)(\vec{x};\vec{a}) = (F(\vec{x};\vec{a}), G(\vec{x};\vec{a})).$$

Now we can give the definition of partial recursive functionals built from basic functionals by the above listed operators.

Definition 9 *The class of partial recursive functionals,* PRECF, *is given by the function algebra:*

$$\mathrm{PRECF} = [\mathcal{Z}^k, \mathcal{S}^k, \mathcal{U}_j^{k,m}, \mathcal{O}_i^{k,m}; \mathcal{C}, \mathcal{R}, \mathcal{V}, \mu].$$

The important property of this definition is the fact, that PRECF can be described by the well-known, intuitive notion of Turing machine.

Proposition 10 *A function $f : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^n$ is in* PRECF *if and only if there is a Turing machine with $k+m+n$ tapes, which behaves in the following way: if we take the binary expansion of $x_1, \ldots, x_k$ and write it in the first k tapes (this expansion might be infinite), write the numbers $a_1, \ldots, a_m$ in each of the following m tapes and begin the computation then: for $f(\vec{x};\vec{a})$ defined – the Turing machine will halt after a finite number of steps and print $(f(\vec{x};\vec{a}))_1, \ldots, (f(\vec{x};\vec{a}))_n$ in the last n tapes; for $f(\vec{x};\vec{a})$ undefined – the machine will not halt.*

The below result presents the interesting fact that classical computability notion of partial recursive functionals has some connection with real recursive functions.

Theorem 11 *If a functional $F : \mathbb{R}^k \times \mathbb{N}^m \to \mathbb{N}^n$ is in* PRECF *then there is a partial recursive function $\tilde{F} : \mathbb{N}^{k+m} \to \mathbb{N}^n$ with the property that $F(\vec{x};\vec{a}) \simeq \vec{b}$ if and only if there is an $m \in \mathbb{N}$ such that $\tilde{F}(;\vec{x} \restriction_n^{\mathrm{I}} \mathrm{N}, \vec{a}) \simeq \vec{b}$ for all natural $n \geq m$.*

Now let us focus on a different side of computability with real recursive functions. What properties of this class can be computed by real recursive functions? To answer this question we should start with some representation of functions by numbers.

Because the set of real recursive functions is enumerable we can find relation between natural numbers and functions from $\mathrm{REC}(\mathbb{R})$. The usual method uses descriptions, i.e. the words which are built in such a way, that all operators and basic functions used to define some function are represented in these words.

Definition 12 *The set of descriptions of* $\mathrm{REC}(\mathbb{R})$ *functions is inductively defined as follows:*

- *u_j^n is a n-ary description of U_j^n, $1 \leq j \leq n \in \mathbb{N}$;*
- *1^n is a n-ary description of $f(x_1, \ldots, x_n) = 1$, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$;*
- *$\bar{1}^n$ is a n-ary description of $f(x_1, \ldots, x_n) = -1$, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$;*
- *0^n is a n-ary description of $f(x_1, \ldots, x_n) = 0$, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$;*
- *if $\langle h \rangle = \langle h_1, \ldots, h_m \rangle$ is a k-ary description of the $\mathrm{REC}(\mathbb{R})$ function h and $\langle g \rangle = \langle g_1, \ldots, g_k \rangle$ is a n-ary description of the $\mathrm{REC}(\mathbb{R})$ function g, then $c(\langle h \rangle, \langle g \rangle)$ is a n-ary description of the composition of h and g;*
- *if $\langle h \rangle = \langle h_1, \ldots, h_n \rangle$ is a k-ary description of the $\mathrm{REC}(\mathbb{R})$ function h and $\langle g \rangle = \langle g_1, \ldots, g_n \rangle$ is a $(k+n+1)$-ary description of the real recursive vector g, then $r(\langle h \rangle, \langle g \rangle)$ is a $(k+1)$-ary description of the function defined by differential recursion;*
- *if $\langle h \rangle = \langle h_1, \ldots, h_m \rangle$ is a $(n+1)$-ary description of the $\mathrm{REC}(\mathbb{R})$ function h, then $ls(\langle h \rangle)$ is a n-ary description of the taking of infinite supremum limit of h;*
- *if $\langle f_1 \rangle, \ldots, \langle f_m \rangle$ are n-ary descriptions of real recursive k-ary scalars $f_1, \ldots, f_m$, then $v(\langle f_1 \rangle, \ldots, \langle f_m \rangle)$ is a k-ary description of the $\mathrm{REC}(\mathbb{R})$ function $f = (f_1, \ldots, f_m)$.*

Let us observe that there are effective methods of enumeration of all descriptions, hence there is one-to-one correspondence between natural numbers and descriptions. Now we can answer some questions about properties of real recursive functions (treated as sets of their descriptions) and analyze whether such properties can be decided by real recursive characteristic functions.

Definition 13 *A real recursive function $\Psi : \mathbb{R}^{m+1} \to \mathbb{R}^n$ is called universal if for every $e \in \mathbb{N}$, $\vec{x} \in \mathbb{R}^m$, we have*

$$\Psi(e,\vec{x}) \simeq f(\vec{x})$$

whenever e is a code of a description of an m-ary function f with n components.

Theorem 14 *There is no universal real recursive function.*

Proof: A diagonal argument will be give us *reductio ad absurdum*. For clarity we present only the case when $m = n = 1$, but the argument may easily be extended. Suppose that there was a universal real recursive function $\Psi : \mathbb{R}^2 \to \mathbb{R}$. We could find its real recursive totalization χ_Ψ and τ_Ψ[1] and the function given by

$$\begin{aligned} g(x) &= \log(1 - \chi_\Psi(x, x)) \\ &= \log(1 - \chi_{\phi_x}(x)) \\ &= \begin{cases} 1 & x \notin Dom(\phi_x), \\ \bot & \text{otherwise;} \end{cases} \end{aligned}$$

would be a real recursive function of arity 1. So let e be a code of g. We have that $e \in Dom(g)$ if and only if $e \notin Dom(\phi_e)$, which is the contradiction we sought. □

In the similar manner we can define and analyze many other problems. For example, problem of checking whether some given description of a real recursive function is minimal for this function cannot be decided by any real recursive characteristic function. In the same way problems of identity of two functions form $\mathrm{REC}(\mathbb{R})$ given by their description or checking whether some $x \in \mathbb{R}$ belongs to the domain of a function from $\mathrm{REC}(\mathbb{R})$ given by its description are undecidable by real recursive functions.

4 Analytical hierarchy

The analytical hierarchy is a hierarchy of predicates over real and natural numbers. It is used extensively in the field of descriptive set theory and the study of recursion on higher types.

A predicate P over real and natural numbers is called recursive if there is a partial recursive functional F such that $F(\vec{x}; \vec{a}) = \begin{cases} 1 & P(\vec{x}, \vec{a}) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$
A predicate Q over real and natural numbers is called arithmetical if it is given using natural number quantifiers over a recursive predicate, i.e., if for some recursive predicate P, $Q(\vec{x}, \vec{a}) \iff (\forall b_1)(\exists b_2) \ldots (\forall b_{n-1})(\exists b_n) P(\vec{x}, \vec{a}, \vec{b})$.

Definition 15 *The analytical hierarchy of predicates consists of three families of predicates over real and natural numbers:*

1. *Σ^1_0 is the class of arithmetical predicates, and $\Pi^1_0 = \Sigma^1_0$.*
2. *Σ^1_{n+1} is the class of predicates given by $\exists y\ P(\vec{x}, y, \vec{a})$, with P in Π^1_n.*
3. *Π^1_{n+1} is the class of predicates given by $\forall y\ P(\vec{x}, y, \vec{a})$, with P in Σ^1_n.*
4. *$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.*

[1]The precise argument of this statement can be found in (4).

We write Δ^1_ω to stand for $\cup_{n \in \mathbb{N}} \Delta^1_n$, which is exactly the set of all analytical predicates. Now we can use the notion of the graph of a functions to place functions within the analytical hierarchy.

Definition 16 *We say that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is in Σ^1_k (Π^1_k, Δ^1_k) if its graph is in Σ^1_k (Π^1_k, Δ^1_k).*

We know that quantifiers may be used to express a rich variety of mathematical ideas, and so we expect that there are many functions in the analytical hierarchy.

Proposition 17 *The functions $1^n, \bar{1}^n, 0^n, \mathrm{U}^n_i, +, \times, x^y, |\cdot|$ and $\lfloor\cdot\rfloor$, as well as the predicates of equality and inequality over the reals, are in Δ^1_0.*

It is not difficult to prove by induction on structure of descriptions of functions that all real recursive functions belong to the analytical hierarchy. However the more fundamental theorem holds, which will be the final stage of this paper.

Theorem 18 $\mathrm{REC}(\mathbb{R})$ *is the class of functions with graphs in the analytical hierarchy, i.e.,* $\mathrm{REC}(\mathbb{R}) = \{f :$ *the predicate given by* $\vec{z} = f(\vec{x})$ *is in* $\Delta^1_\omega\}$.

We will describe briefly the method of the proof that every function in Δ^1_ω is real recursive. As the first step it is sufficient to prove that every predicate in the analytical hierarchy has a real recursive characteristic. It is done by showing that the characteristic function of every predicate $P \in \Pi^1_1$ has a real recursive extension and then this result can be extended to the conclusion that the characteristic of every predicate P in the analytical hierarchy is real recursive. The last step in the proof is the proposition that if the graph of a function has a real recursive characteristic then the function itself is real recursive, which is below presented for scalar functions.

Proposition 19 *Let χ_f denote the characteristic function of the graph of $f : \mathbb{R}^m \to \mathbb{R}$, i.e.,*

$$\chi_f(z, \vec{x}) = \begin{cases} 1 & z = f(\vec{x}) \\ 0 & \textit{otherwise.} \end{cases}$$

If χ_f is real recursive, then so is f.

Proof: We construct a search operator, somewhat like minimalization, but with the whole $\mathbb{R}$ as search domain. Considering the function $\sigma(x) = \frac{e^x}{1+e^x}$ and its inverse $\sigma^{-1}(y) = \log(y) - \log(1-y)$. The function σ surjectively maps $\mathbb{R}$ into $(0,1)$. So let

$$\begin{aligned} F(\vec{x}, z) &= (1 - \chi_f(z, \vec{x})) + \chi_f(z, \vec{x})\sigma(z) \\ &= \begin{cases} \sigma(z) & z = f(\vec{x}), \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Now we can compute f in the following way.

$$f(\vec{x}) = \sigma^{-1}(\mathbf{Inf}(F)(\vec{x})).$$

where $\mathbf{Inf}(F)$ is a real recursive operator[2] given as $\mathbf{Inf}(F)(\vec{x}) = \inf_{y \in \mathbb{R}} F(\vec{x}, y)$. □

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[2]The proof that $\mathrm{REC}(\mathbb{R})$ is closed for $\mathbf{Inf}$ can be found in (4).